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Haar Wavelet Method for Constrained Nonlinear Optimal Control Problems with Application to Production Inventory Model

(Kaedah Gelombang Kecil Haar untuk Masalah Kawalan Optimum Kekangan tak Linear dengan Model Aplikasi untuk Inventori Pengeluaran)

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ABSTRACT

A new numerical method was proposed in this paper to address the nonlinear quadratic optimal control problems, with state and control inequality constraints. This method used the quasilinearization technique and Haar wavelet operational matrix to convert the nonlinear optimal control problem into a sequence of quadratic programming problems. The inequality constraints for trajectory variables were transformed into quadratic programming constraints using the Haar wavelet collocation method. The proposed method was applied to optimize the control of the multi-item inventory model with linear demand rates. By enhancing the resolution of the Haar wavelet, we can improve the accuracy of the states, controls and cost. Simulation results were also compared with other researchers' work.

Keywords: Direct method; Haar wavelet operational matrix; optimal control; quadratic programming problem

ABSTRAK

Kaedah berangka baru telah dicadangkan dalam kertas ini untuk menangani masalah kawalan optimum quadratik tak linear dengan kekangan keadaan serta kawalan ketidaksamaan. Kaedah ini menggunakan teknik quasipelinearan dan matriks operasi gelombang kecil Haar untuk menukar masalah kawalan optimum tak linear kepada suatu turutan masalah pengaturcaraan quadratik. Kekangan ketidaksamaan bagi pemboleh ubah trajektori diubah menjadi kekangan pengaturcaraan quadratik menggunakan kaedah kolokasi gelombang kecil Haar. Kaedah cadangan telah digunakan untuk mengoptimumkan kawalan model inventori item berbilang dengan kadar permintaan linear. Dengan mempertingkatkan resolusi gelombang kecil Haar, ketepatan keadaan, kawalan serta kos boleh ditambah baik. Keputusan simulasi juga dibandingkan dengan hasil penyelidikan lain.

Kata kunci: Kawalan optimum; kaedah langsung; masalah pengaturcaraan quadratik; matriks operasi gelombang kecil Haar

INTRODUCTION

Optimal control problems without constraints can be solved successfully using most of the direct and indirect techniques. However, inequality constraints often generate both analytical and computational difficulties. Thus, researchers aim to solve constrained optimal control problems with numerical methods. The direct method is widely used to solve nonlinear optimal control problems. It obtains an optimal solution by directly minimizing the constrained performance index. Furthermore, this method converts the optimal control problem into a mathematical programming problem using either the discretization or the parameterization technique. Parameterization methods are classified into three types: state, control, and state control.

Many researchers have studied the theoretical aspects of the inequality constraints of trajectory. Mehra and Davis (1972) noted that the complications in handling trajectory inequality constraints in gradient or conjugate gradient methods were caused by the exclusive use of control variables as independent variables in the search procedure. In response, they presented the so-called generalized gradient technique. Jaddu (1998) established some numerical methods based on a parameterization technique with Chebyshev polynomials to solve unconstrained and constrained optimal control problems using the quasilinearization method. Jaddu (2002) later extended this concept to nonlinear optimal control problems with terminal state and control inequality constraints, as well as to simple bounds on state variables.

Historically, orthogonal functions have been used to solve various problems of dynamic systems. These functions mainly convert underlying differential equations into integral equations through integration, approximate the various functions in the equation by truncating the orthogonal series and eliminate integral operations through the operational matrix of integration. Thus, they reduce the original problems to those of solving a system of linear algebraic equations. A typical example is the Haar wavelet function, which possesses useful properties such as orthogonality, compact support and the capability to represent functions at different levels of resolution. It has been applied to a wide range of application such as in system analysis (Chen & Hsiao 1999) and numerical solutions of nonlinear integral equations (Aziz & Siraj 2013), boundary-value problems (Siraj et al. 2011, 2010) and optimal control problems (Swaidan & Hussin 2013). Solving optimal control problem through orthogonal functions, especially Haar wavelets, is an active research area. In fact, Dai and Cochran (2009) converted optimal control problems into nonlinear programming (NLP) parameters at the collocation points using a Haar wavelet technique. NLP problems can be solved using NLP solvers such as the SNOPT. Han and Li (2011) also presented a numerical method to solve nonlinear optimal control problems with terminal state and state and control inequality constraints. This method is based on quasilinearization and Haar functions. Moreover, the researchers parameterized only the state variables and added artificial controls to equalize the number of state and control. In the present study, we do not incorporate artificial variables, but we parameterize both states and control variables. In addition, Marzban and Razzaghi (2010) presented a numerical method to address constrained and nonlinear optimal control problems. Although their method is also based on Haar wavelets but it required a set of necessary conditions. Our method is much easier to implement than that of Dai and Cochran (2009) and Han and Li (2011) because our method does not require time transformation to the domain [0,1].

We derived a novel method to solve nonlinear unconstrained optimal control problems using indirect method from a recent study (Swaidan & Hussin 2013) and our present study aims to effectively compute for optimal control using direct method. We introduced a numerical method to solve nonlinear optimal control problems under inequality constraints. We parameterize both the states and the controls using Haar wavelet functions. The nonlinear optimal control problem is converted into a sequence quadratic programming problems through the quasilinearization iterative technique. Moreover, the inequality constraints for trajectory variables are transformed into quadratic programming constraints using the Haar wavelet collocation method.

The paper is organized as follows: The second section describes the problem statement, which involves constrained nonlinear optimal control problems. The next section formulates the Haar wavelet system and the Haar wavelet operational matrices required in subsequent development. The section that follows presents the proposed method to approximate the solution to the constrained nonlinear optimal control problem using Haar wavelet collocation and quasilinearization methods. In the final section, the proposed method is applied to solve the optimal control of a two-item inventory model with stock-dependent deterioration and linear demand rates.

Problem Statement

The control system is given by differential equation in the form

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(0) = \mathbf{x}_0, \; 0 \le t \le t_f, \tag{1}$$

where, $x(t) \in \mathrm{R}^{n_1}$ is the state vector; $u(t) \in \mathrm{R}^{n_2}$ is the control vector; f is continuously differentiable with respect to all its arguments; $\mathbf{x}_0$ is the initial condition vector and t_f is a known finite time and they are subjected to the following constraints:

$$x(t) \le \mathbf{x}_{max}, \; x(t) \ge \mathbf{x}_{min}, \tag{2}$$

$$u(t) \le \mathbf{u}_{max}, \quad u(t) \ge \mathbf{u}_{min}. \tag{3}$$

The vector inequalities such as $x(t) \le \mathbf{x}_{max}$ means $x_i(t) \le \mathbf{x}_{max,i}$ for all $i = 1,2,\ldots,n_1$.

We aim to determine the optimal control $u^*(t)$ and the corresponding state vector $x(t)$ that minimizes the following performance index:

$$J = \int_0^{t_f} (x^T(t)\, Q\, x(t) + u^T(t)\, R\, u(t))\, dt \tag{4}$$

where, $Q \in \mathrm{R}^{n_1 \times n_1}$ is a positive semi definite matrix and $R \in \mathrm{R}^{n_2 \times n_2}$ is a positive definite matrix. It is assumed that the problem in (1)-(4) has a unique solution.

Haar Wavelets

The orthogonal set of the Haar wavelets $h_i(x)$ is a group of square waves over the interval $[\tau_1, \tau_2)$, which is defined as follows:

$$h_1(x) = \begin{cases} 1, & \tau_1 \le x < \frac{1}{2}(\tau_1 + \tau_2), \\ -1, & \frac{1}{2}(\tau_1 + \tau_2) \le x \le \tau_2, \\ 0, & \textbf{elsewhere}. \end{cases} \tag{5}$$

Other wavelets can be obtained by dilating and translating the mother wavelet $h_1(x)$. In general, $h_i(x) = h_1(2^j x - k)$, where $i = 2^j + k$, $i, j \in N \cup \{0\}$, and $0 \le k < 2^j$ and satisfies

$$\int_{\tau_1}^{\tau_2} h_i(x) h_r(x)\, dx = \begin{cases} (\tau_2 - \tau_1) 2^{-j}, & i = r \\ 0, & i \ne r \end{cases}. \tag{6}$$

Any $f(x) \in L^2([\tau_1, \tau_2])$can be expanded into a Haar series of infinite terms:

$$f(x) = c_0 h_0(x) + c_1 h_1(x) + c_2 h_2(x) + \ldots, \tag{7}$$

where h_0 is the characteristic function $\chi_{[\tau_1,\tau_2)}$. If $f(x)$ is approximated as a piecewise constants, then the decomposition can be terminated as follows:

$$f_m(x) = \sum_{i=0}^{m-1} c_i h_i(x) \tag{8}$$

where $i = 2^j + k$, $j = 0,1,2,\ldots \log_2 m - 1$ and $k = 0,1,2,\ldots, 2^j - 1$ and m is the dyadic resolution. The Haar coefficients that are

$$c_i = \frac{2^j}{\tau_2 - \tau_1} \int_{\tau_1}^{\tau_2} f(x) h_i(x) dx, \tag{9}$$

can be obtained by minimizing the integral square error $\int_{\tau_1}^{\tau_1} (f(x) - \sum_{i=0}^{m-1} c_i h_i(x))^2 dx$. The sum in (8) can be compacted into the form:

$$f(x) = \mathbf{c}_m^T \mathbf{h}_m(x), \tag{10}$$

where, $\mathbf{c}_m = [c_0 \ c_1 \ \dots \ c_{m-1}]^T$ is the coefficient vector and $\mathbf{h}_m(x) = [h_0(x) \ \ h_1(x) \ \dots \ h_{m-1}(x)]^T$, is the Haar function vector. From (6) we obtain

$$\int_{\tau_1}^{\tau_2} \mathbf{h}_m(x) \mathbf{h}_m^T(x) dx = E_m, \tag{11}$$

where $E_m = (\tau_2 - \tau_1) diag([1 \ \underbrace{2^0}_{2^0 \, time} \ \underbrace{2^{-1} 2^{-1}}_{2^1 \, times} \dots \underbrace{2^{-j} 2^{-j} \dots 2^{-j}}_{2^j \, times} \dots])$, for $j = 0,1,2,\dots,\log_2 m - 1$.

At collocation points $x_r = (\tau_1 + \frac{\tau_2 - \tau_1}{2m}(2r-1))$, $r = 1, 2, 3, \dots, m{-}1$, the Haar function vector $\mathbf{h}_m(x)$ can be expressed in the matrix form $\mathbf{H}_m$, the elements of which are given by:

$$(\mathbf{H}_m)_{i,r} = h_i(x_r). \tag{12}$$

For instance, the fourth Haar wavelet matrix $\mathbf{H}_4$ can be represented in matrix form as:

$$\mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The integration of $h_i(x)$ into the particular interval of $[0, \tau)$ can be expanded into a Haar series as in Chen and Hsiao (1997).

$$\int_0^x \mathbf{h}_m(x) dx \cong \mathbf{P}_m \mathbf{h}_m(x), \tag{13}$$

where the $m \times m$ matrix $\mathbf{P}_m$ is the operational matrix of integration, which is recursively obtained as follows (Swaidan & Hussin 2013):

$$\mathbf{P}_m = \frac{1}{2m} \begin{bmatrix} 2m\mathbf{P}_{m/2} & -\tau \mathbf{H}_{m/2} \\ -\tau \mathbf{H}_{m/2}^{-1} & O_{m/2} \end{bmatrix} \ , \ \mathbf{P}_1 = \left[\frac{\tau}{2}\right]. \tag{14}$$

In order to simplify the product of the two functions $f(x) = \mathbf{c}^T \mathbf{h}(x)$ and $g(x) = \mathbf{d}^T \mathbf{h}(x)$, we must obtain the product of $\mathbf{h}(x)$ and $\mathbf{h}^T(x)$. The product can be expanded into a Haar series with a Haar coefficient matrix $\mathbf{M}_m$ as:

$$f(x)g(x) = \mathbf{d}^T \mathbf{h}_m(x) \mathbf{h}_m^T(x) \ \mathbf{c} = \mathbf{d}^T \mathbf{M}_m(\mathbf{c}) \mathbf{h}(x). \tag{15}$$

where, $\mathbf{M}_m$ is an $m \times m$ matrix that is referred to as the product operational matrix. It was first expressed by Hsiao and Wu (2007), as follows:

$$\mathbf{M}_m(\mathbf{c}) = \begin{bmatrix} \mathbf{M}_{m/2} & \mathbf{H}_{m/2} diag(\mathbf{c}_b) \\ diag(\mathbf{c}_b) \mathbf{H}_{m/2}^{-1} & diag(\mathbf{c}_a^T \mathbf{H}_{m/2}) \end{bmatrix}, \tag{16}$$

where, $\mathbf{M}_1 = c_0$ and $\mathbf{c}_a = [c_0, \dots, c_{m/2-1}]^T$, $\mathbf{c}_b = [c_{m/2}, \dots, c_{m-1}]^T$.

In the subsequent sections, we drop the subscript m to economize the notation if it is not confusing.

NUMERICAL SOLUTION TO THE NONLINEAR OPTIMAL CONTROL PROBLEM

We proposed the following numerical solution to a nonlinear optimal control problem with inequality constraints. At each step of this algorithm, we identify an approximate solution to optimal control problems (1)-(4). The orthogonal Haar wavelet is used as a basis to approximate $x(t)$ and control $u(t)$.

By applying the quasilinearization method (Bellman & Kalaba 1965), we can therefore replace optimal control problem (1)-(4) with the following sequence of constrained linear-quadratic optimal control problems:

Minimizes

$$J^{[k]} = \int_0^{t_f} (x^{[k]T}(t) Q x^{[k]}(t) + u^{[k]T}(t) R u^{[k]}(t)) dt, \tag{17}$$

subject to the linearized time varying state equations:

$$\frac{dx^{[k]}(t)}{dt} = A^{[k-1]}(t) x^{[k]} + B^{[k-1]}(t) u^{[k]}, x^{[k]}(0) = \mathbf{x}_0, \ k \geq 1, \tag{18}$$

where

$$A^{[k]}(t) = \frac{\partial f(x,u,t)}{\partial x}\Big|_{x^k, u^k}, \tag{19}$$

$$B^{[k]}(t) = \frac{\partial f(x,u,t)}{\partial u}\Big|_{x^k, u^k}, \tag{20}$$

are the $n_1 \times n_1$ and $n_1 \times n_2$ matrices, respectively and the inequality constraints are expressed as follows:

$$x(t)^{[k]} \leq \mathbf{x}_{max}, \ x(t)^{[k]} \geq \mathbf{x}_{min}, \tag{21}$$

$$u(t)^{[k]} \leq \mathbf{u}_{max}, \ u(t)^{[k]} \geq \mathbf{u}_{min}, \tag{22}$$

The initial matrices $A^0(t)$ and $B^0(t)$ were determined using an approximately accurate initial assumption of $x^0(t)$ and $u^0(t)$ which does not cause the algorithm to diverge. We suggest starting from the initial condition vector $\mathbf{x}_0$.

OPTIMAL CONTROL PROBLEM USING HAAR WAVELET METHOD

The optimal control problem described in (17)-(22) is formalized using the orthogonal functions described in the third section. Given that Haar wavelet functions are not continuous, we first expand $x(t)$ as follows:

$$\dot{x}(t) = \mathbf{c}^T \mathbf{h}(t), \tag{23}$$

where $\mathbf{c}^T$ is now an unknown $n_1 \times m$ coefficient matrix.

Integrating (23) and by applying (13), we obtain

$$x(t) = \int_0^t \mathbf{c}^T \mathbf{h}(t)dt + \mathbf{x}_0, \quad (24)$$

$$x(t) = \mathbf{c}^T \mathbf{P}\, \mathbf{h}(t) + \mathbf{x}_0\theta^T \mathbf{h}(t), \quad (25)$$

where $\mathbf{x}_0$ is the n_1 column vector and $\theta^T = [1,0,0,\ldots,0]$ is an m row vector. However, for the controls, we let

$$u(t) = \mathbf{d}^T \mathbf{h}(t), \quad (26)$$

where $\mathbf{d}^T$ is an unknown $n_2 \times m$ coefficient matrix.

Let $vec(A)$ denotes the transformation of stacking the columns of A^T and $\otimes$ represents the Kroneker product operation. Equations (23), (25) and (26) can then be written as

$$\dot{x}(t) = \left(I_{n_1} \otimes \mathbf{h}^T(t)\right) vec(\mathbf{c}), \quad (27)$$

$$x(t) = (I_{n_1} \otimes \mathbf{h}^T(t)\mathbf{P}^T) vec(\mathbf{c}) + (I_{n_1} \otimes \mathbf{h}^T(t)) vec(\mathbf{x}_0\theta^T), \quad (28)$$

$$u(t) = (I_{n_2} \otimes \mathbf{h}^T(t)) vec(\mathbf{d}), \quad (29)$$

where I_{n_1} and I_{n_2} denotes $n_1 \times n_1$ and $n_2 \times n_2$ identity matrices, respectively.

APPROXIMATION OF THE PERFORMANCE INDEX

When (28) and (29) are substituted into (17) and simplified, we obtain:

$$\begin{aligned} J = \int_0^{t_f} \{ & vec^T(\mathbf{c})(I_{n_1} \otimes \mathbf{P}\,\mathbf{h}(t)) Q (I_{n_1} \otimes \mathbf{h}^T(t)\mathbf{P}^T) vec(\mathbf{c}) \\ & + vec^T(\mathbf{c})(I_{n_1} \otimes \mathbf{P}\,\mathbf{h}(t)) Q (I_{n_1} \otimes \mathbf{h}^T(t)) vec(\mathbf{x}_0\theta^T) \\ & + vec^T(\mathbf{x}_0\theta^T)(I_{n_1} \otimes \mathbf{P}\,\mathbf{h}(t)) Q (I_{n_1} \otimes \mathbf{h}^T(t)\mathbf{P}^T) vec(\mathbf{c}) \\ & + vec^T(\mathbf{x}_0\theta^T)(I_{n_1} \otimes \mathbf{p}\,\mathbf{h}(t)) Q (I_{n_1} \otimes \mathbf{h}^T(t)\mathbf{P}^T) vec(\mathbf{x}_0\theta^T) \\ & + vec^T(\mathbf{d})(I_{n_2} \otimes \mathbf{h}(t)) R (I_{n_2} \otimes \mathbf{h}^T(t)) vec(\mathbf{d}) \} dt. \end{aligned} \quad (30)$$

By utilizing Kronecker product properties (Brewer 1978) and (11), we determine

$$\begin{aligned} J = & vec^T(\mathbf{c})(Q \otimes \mathbf{P}E\mathbf{P}^T) vec(\mathbf{c}) + vec^T(\mathbf{c})(Q \otimes \mathbf{P}E) vec(\mathbf{x}_0\theta^T) \\ & + vec^T(\mathbf{x}_0\theta^T)(Q \otimes E\mathbf{P}^T)) vec(\mathbf{c}) + vec^T(\mathbf{x}_0\theta^T)(Q \otimes E) vec(\mathbf{x}_0\theta^T) \\ & + vec^T(\mathbf{d})(R \otimes E) vec(\mathbf{d}). \end{aligned} \quad (31)$$

The performance index can be rewritten in quadratic form as follows (Bhatti 2000):

$$J = \frac{1}{2} \mathbf{Z}^T \mathbf{H}_{ess} \mathbf{Z}Z + \mathbf{F}^T \mathbf{Z} + \mathbf{e}, \quad (32)$$

where

$$\mathbf{Z} = [vec^T(\mathbf{c}) \quad vec^T(\mathbf{d})]^T, \quad (33)$$

$$\mathbf{H}_{ess} = \begin{bmatrix} Q \otimes \mathbf{P}E\mathbf{P}^T & O \\ O & R \otimes E \end{bmatrix}, \quad (34)$$

$$\mathbf{F} = [2vec^T(\mathbf{x}_0\theta^T)(Q \otimes E\,\mathbf{P}^T) \quad O]^T, \quad (35)$$

$$\mathbf{e} = [vec^T(\mathbf{x}_0\theta^T)(Q \otimes E) vec(\mathbf{x}_0\theta^T)], \quad (36)$$

are $m(n_1 + n_2) \times 1$, $m(n_1 + n_2) \times m\,(n_1 + n_2)$, $m(n_1 + n_2) \times 1$ and 1×1 matrices, respectively.

APPROXIMATIONS OF SYSTEM DYNAMICS AND INEQUALITY CONSTRAINTS

State equations are approximated in terms of the unknown coefficients of state and control variables by substituting (27), (28) and (29) into (18). Once these equations were simplified, the time varying matrices *A(t)* and *B(t)* should be expressed in terms of the Haar wavelets.

The function of $(i,j)^{th}$ element of $A(t)$ can be approximated using (10) as:

$$[A(t)]_{ij} = G_{ij}^T h(t), \quad (37)$$

where $G_{ij}^T = \left[g_1^{ij} g_2^{ij} \ldots g_m^{ij}\right]$ is the m row vector of the known coefficients of the Haar wavelet function for each $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, n_1$.

Similarly, the elements of *B(t)* can be expanded using Haar wavelet function:

$$[B(t)]_{ij} = L_{ij}^T \mathbf{h}(t), \quad (38)$$

where L_{ij}^T is the constant $1 \times m$ row coefficients of a Haar wavelet function for and $j = 1, 2, \ldots, n_2$.

Then (37) and (38) can be rewritten in compact form by using Kronecker product properties as in Brewer (1978):

$$A(t) = \mathbf{G}^T (I_{n_1} \otimes \mathbf{h}(t)), \quad (39)$$

$$B(t) = \mathbf{L}^T (I_{n_2} \otimes \mathbf{h}(t)), \quad (40)$$

where the block matrices $\mathbf{G}^T = \begin{bmatrix} G_{11}^T & G_{12}^T & \cdots & G_{1n_1}^T \\ \vdots & \vdots & \vdots & \vdots \\ G_{n_11}^T & G_{n_12}^T & \cdots & G_{n_1n_1}^T \end{bmatrix}$ and $\mathbf{L}^T = \begin{bmatrix} L_{11}^T & L_{12}^T & \cdots & L_{1n_2}^T \\ \vdots & \vdots & \vdots & \vdots \\ L_{n_11}^T & L_{n_12}^T & \cdots & L_{n_1n_2}^T \end{bmatrix}$, are of sizes $n_1 \times n_1 m$ and $n_1 \times n_2 m$, respectively.

Given the notation above, substituting the transpose of (27), (28), (29) with (39) and (40) into (18) and simplified using (15) to obtain

$$\begin{aligned} & vec^T(\mathbf{c}) - vec^T(\mathbf{c})(I_{n_1} \otimes \mathbf{P})\hat{\mathbf{M}}(\mathbf{G}) - \\ & vec^T(\mathbf{d})\hat{\mathbf{M}}(\mathbf{L}) = vec^T(\mathbf{x}_0\theta^T)\hat{\mathbf{M}}(\mathbf{G}), \end{aligned} \quad (41)$$

where $\hat{\mathbf{M}}(\mathbf{G})$ is $n_1m\times n_1m$ and $\hat{\mathbf{M}}(\mathbf{L})$ is $n_2m\times n_1m$. For example,

$$\hat{\mathbf{M}}(\mathbf{G})=\begin{bmatrix}\mathbf{M}(G_{11}) & \mathbf{M}(G_{21}) & \dots & \mathbf{M}(G_{n_11})\\ \vdots & \vdots & \vdots & \vdots\\ \mathbf{M}(G_{1n_1}) & \mathbf{M}(G_{2_{n_1}}) & \dots & \mathbf{M}(G_{n_1n_1})\end{bmatrix}. \quad (42)$$

When (41) is transformed into standard system of linear equation form, we get

$$\begin{bmatrix} I_{n_1m}-\hat{\mathbf{M}}^T(\mathbf{G})(I_{n_1}\otimes \mathrm{P}^T) & -\hat{\mathbf{M}}^T(\mathbf{L})\end{bmatrix}\begin{bmatrix} vec(\mathbf{c})\\ vec(\mathbf{d})\end{bmatrix}=$$

$$\begin{bmatrix}\hat{\mathbf{M}}^T(\mathbf{G})vec(\mathbf{x}_0\theta^T)\end{bmatrix}. \quad (43)$$

Note that in this equation all the multiplications must be performed block wise as in Lancaster and Tismenetsky (1985).

The inequality constraints for state and control variables should also be determined in this study. These constraints were converted into quadratic programming constraints using the Haar wavelet collocation method. By substituting (28) and (29) into (21) and (22), respectively, at collocation points, we establish:

$$(I_{n_1}\otimes \mathbf{H}^T\mathbf{P}^T)vec(\mathbf{c})+vec(\mathbf{x}_0\theta^T\mathbf{H})\le vec(\mathbf{x}_{max}\theta^T\mathbf{H}), \quad (44)$$

$$(I_{n_1}\otimes \mathbf{H}^T\mathbf{P}^T)vec(\mathbf{c})+vec(\mathbf{x}_0\theta^T\mathbf{H})\ge vec(\mathbf{x}_{min}\theta^T\mathbf{H}), \quad (45)$$

$$(I_{n_2}\otimes \mathbf{H}^T)vec(\mathbf{d})\le vec(\mathbf{u}_{max}\theta^T\mathbf{H}), \quad (46)$$

$$(I_{n_2}\otimes \mathbf{H}^T)vec(\mathbf{d})\ge vec(\mathbf{u}_{min}\theta^T\mathbf{H}). \quad (47)$$

By moving the constant vector of (44) and (45) to the other side and by changing the signs of the (45) and (47), we generate:

$$(I_{n_1}\otimes \mathbf{H}^T\mathbf{P}^T)vec(\mathbf{c})\le vec(\mathbf{x}_{max}\theta^T\mathbf{H})-vec(\mathbf{x}_0\theta^T\mathbf{H}), \quad (48)$$

$$-(I_{n_1}\otimes \mathbf{H}^T\mathbf{P}^T)vec(\mathbf{c})\le vec(\mathbf{x}_0\theta^T\mathbf{H})-vec(\mathbf{x}_{min}\theta^T\mathbf{H}), \quad (49)$$

$$(I_{n_2}\otimes \mathbf{H}^T)vec(\mathbf{d})\le vec(\mathbf{u}_{max}\theta^T\mathbf{H}), \quad (50)$$

$$-(I_{n_2}\otimes \mathbf{H}^T)vec(\mathbf{d})\le -vec(\mathbf{u}_{min}\theta^T\mathbf{H}). \quad (51)$$

By combining (48)-(51) after adding zeros of the missing variables in these equations, we obtain the following form of inequality constraints,

$$\begin{bmatrix}(I_{n_1}\otimes\mathbf{H}^T\mathbf{P}^T) & O_{n_1m\times n_2m}\\ -(I_{n_1}\otimes\mathbf{H}^T\mathbf{P}^T) & O_{n_1m\times n_2m}\\ O_{n_1m\times n_1m} & (I_{n_2}\otimes\mathbf{H}^T)\\ O_{n_1m\times n_1m} & -(I_{n_2}\otimes\mathbf{H}^T)\end{bmatrix}\begin{bmatrix}vec(\mathbf{c})\\ vec(\mathbf{d})\end{bmatrix}\le\begin{bmatrix}vec(\mathbf{x}_{max}\theta^T\mathbf{H})-vec(\mathbf{x}_0\theta^T\mathbf{H}\\ vec(\mathbf{x}_0\theta^T\mathbf{H})-vec(\mathbf{x}_{min}\theta^T\mathbf{H}\\ vec(\mathbf{x}_{max}\theta^T\mathbf{H})\\ -vec(\mathbf{u}_{min}\theta^T\mathbf{H})\end{bmatrix}. \quad (52)$$

Based on the previous reformulation, optimal control problems (17)-(22) can be approximated by the following quadratic programming problem (Bhatti 2000):

$$min\frac{1}{2}\mathbf{Z}^T\mathbf{H}_{ess}\mathbf{Z}+\mathbf{F}^T\mathbf{Z}+\mathbf{e}, \quad (53)$$

subject to

$$\mathbf{F}_1\mathbf{Z}=\mathbf{b}_1 \quad (54)$$

$$\mathbf{F}_2\mathbf{Z}\le\mathbf{b}_2 \quad (55)$$

where

$$\mathbf{F}_1=\lfloor I_{n1m}-\hat{\mathbf{M}}^T(\mathbf{G})(I_{n1}\otimes\mathbf{P}^T)-\ \hat{\mathbf{M}}^T(\mathbf{L})\rfloor, \quad (56)$$

$$\mathbf{b}_1=\lfloor\ \hat{\mathbf{M}}^T(\mathbf{G})vec(\mathbf{x}_0\theta^T)\rfloor, \quad (57)$$

$$\mathbf{F}_2=\begin{bmatrix}(I_{n_1}\otimes\mathbf{H}^T\mathbf{P}^T) & O_{n_1m\times n_2m}\\ -(I_{n_1}\otimes\mathbf{H}^T\mathbf{P}^T) & O_{n_1m\times n_2m}\\ O_{n_1m\times n_1m} & (I_{n_2}\otimes\mathbf{H}^T)\\ O_{n_1m\times n_1m} & -(I_{n_2}\otimes\mathbf{H}^T)\end{bmatrix}, \quad (58)$$

$$\mathbf{b}_2=\begin{bmatrix}vec(\mathbf{x}_{max}\theta^T\mathbf{H})-vec(\mathbf{x}_0\theta^T\mathbf{H})\\ vec(\mathbf{x}_0\theta^T\mathbf{H})-vec(\mathbf{x}_{min}\theta^T\mathbf{H})\\ vec(\mathbf{u}_{max}\theta^T\mathbf{H})\\ -vec(\mathbf{u}_{min}\theta^T\mathbf{H})\end{bmatrix}. \quad (59)$$

Equations (53)-(59) represent a standard quadratic programming problem which can be solved using solver such as quadprog in MATLAB. Once we obtain the optimal solution to the unknown parameters $\mathbf{Z}$, we substitute these parameters into (25) and (26) to determine the new nominal states $x^k(t)$ and controllers $u^k(t)$ to be used in the subsequent iteration. These new nominal trajectories should be substituted into (18) to derive the next optimal control problem that is constrained linear quadratic. This procedure should be repeated until an acceptable convergence is achieved. The iteration is terminated when the difference between the two cost functions $|J^{k+1}-J^k|$ is sufficiently small.

APPLICATION OF HAAR WAVELET METHOD TO PRODUCTION-INVENTORY MODEL

Although most inventory models generally deal with a single-item model and time-varying demand rates with finite time horizon (Balkhi & Benkherouf 2004; Omar 2012), however, such models are seldom applied in the real world. Multi-item inventory models are more realistic than single-item models. In multi-item models, the second item in an inventory favors the demand for the first and vice-versa. Here, we consider a factory that produces two items and has a finished goods warehouse. The objective function includes the sum of inventory holding costs, the holding costs of one item as a result of the presence of others and the production costs. The problem is regarded as an optimal control problem with two state and two control variables, which are inventory levels y_i and production rates v_i, respectively (El-Gohary & Elsayed 2008; Sethi & Thompson 2006). For $i = 1,2,$, let c_{ii} and h_{ii} be the production cost coefficients and the holding cost coefficients, respectively and h_{12} be the inventory holding cost coefficient of y_1 due to the presence of unit of y_2. Then the total cost

$$J = \int_0^{t_f}\left\{\sum_{i=1}^{2}\left(h_{ii}\left(y_i\right)-\hat{y}_i\right)^2 + c_{ii}\left(v_i-\hat{v}_i\right)^2\right) + 2h_{12}\left(y_1-\hat{y}_1\right)\left(y_2-\hat{y}_2\right)\right\}dt, \tag{60}$$

where $h_{11}, h_{22} > h_{12}^2$, $h_{ii} > 0$, $c_{ii} > 0$ and $\hat{y}_i$ and $\hat{v}_i$ are the inventory levels and production rates goal, respectively.

This cost needs to be minimize, subject to (El-Gohary & Elsayed 2008)

$$\begin{aligned}\dot{y}_1(t) &= -y_1(t)(\theta_1 + a_{12}y_2(t) + a_{11}y_1(t)) - D_1(y_1, y_2, t) + v_1(t)\\ \dot{y}_2(t) &= -y_2(t)(\theta_2 + a_{21}y_1(t) + a_{22}y_2(t)) - D_2(y_1, y_2, t) + v_2(t)\end{aligned} \tag{61}$$

where $D_i(y_1, y_2, t)$ is the demand rates, θ_i is the natural deterioration rates of y_i, a_{ii} is the deterioration coefficient due to self-contact of y_i and a_{ij} is the demand coefficient of y_i due to the presence of unit of y_j, $i \neq j$ with constraints

$$y_i(t) \geq 0, \tag{62}$$

$$v_i(t) \geq 0. \tag{63}$$

This system is nonlinear and is difficult to solve analytically. Therefore, we address it numerically and display the results graphically. The objective function (60) can be economically interpreted as an aim to maintain the inventory levels ($y_1(t)$, $y_2(t)$) and production rates ($v_1(t)$, $v_2(t)$) at values that are approximate to the desired ones. The system dynamics in (61) can be used to describe the time evolution of inventory levels and production rates.

With the substitution:

$$x_i(t) = y_i(t) - \hat{y}_i, \tag{64}$$

$$u_i(t) = v_i(t) - \hat{v}_i, \tag{65}$$

the equations (60)-(63) can be reformulated as follows:

$$J = min\int_0^{t_f}\left(x^T(t)Qx(t) + u^T(t)Ru(t)\right)dt, \tag{66}$$

subject to

$$\begin{aligned}\dot{x}_1(t) &= -(x_1(t) - \hat{y}_1)(\theta_1 + a_{12}(x_2(t) + \hat{y}_2) + a_{11}(x_1(t) + \hat{y}_1))\\ &\quad - D_1(x_1, x_2, t) + u_1(t) + \hat{v}_i,\\ \dot{x}_2(t) &= -(x_2(t) - \hat{y}_2)(\theta_2 + a_{21}(x_1(t) + \hat{y}_1) + a_{22}(x_2(t) + \hat{y}_2))\\ &\quad - D_2(x_1, x_2, t) + u_2(t) + \hat{v}_2,\end{aligned} \tag{67}$$

with constraints

$$x_i(t) + \hat{y}_i \geq 0, \tag{68}$$

$$u_i(t) + \hat{v}_i \geq 0, \tag{69}$$

where $Q = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22}\end{bmatrix}$ and $R = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{22}\end{bmatrix}$

The numerical solution to this problem as obtained using the proposed method with the following parameter values and intial states as in El-Gohary and Elsayed (2008), $h_{11} = 4, h_{12} = -4, h_{22} = 5, c_{11} = 6, c_{22} = 5, a_{11} = 0.04$, $a_{22} = 0.05$, $a_{12} = 0.7$, $a_{21} = 0.6$, $\theta_1 = 0.02$, $\theta_2 = 0.03$, $\hat{y}_1 = 4$, $\hat{y}_2 = 3$, $\hat{v}_1 = 9$, $\hat{v}_2 = 8$, $y_{10} = 2$, $y_{20} = 1$, $D_1 = 3x_1 + 0.6$ and $D_2 = 4x_2 + 0.8$. The horizon of planning time is $t_f = 5$.

Each state and control variable is approximated with Haar wavelet functions at m^{th} resolution. Furthermore, optimal control problem, which is subject to constraints (67)-(69) is solved beginning with nominal trajectories $x_1^0 = -2, x_2^0 = -2$ for $m = 16, 32, 64, 128$ and 256. For each m, convergence is achieved in six quasilinearization iterations. The iteration is terminated when the difference between two cost functions $|J^{k+1} - J^k|$ is less than $\varepsilon = 0.0001$.

Table 1 summarizes the results obtained from these five cases of Haar wavelet resolution, including the simulated optimal values of inventory levels and production rates, as well as the total cost at the end of the planning horizon period. Figures 1-4 show the optimal values of the inventory levels and the production rates for $m = 256$ and it successive quasilinearization iteration. The step functions in Figures 3 and 4 are not visible because the collocation points are too close to each other. Table 1

TABLE 1. Simulation results of application with linear demand rates for m = 16, 32, 64, 128 and 256

m	$y_1^*(t_f)$	$y_2^*(t_f)$	$v_1^*(t_f)$	$v_2^*(t_f)$	J^*
16	2.0816	1.3484	9.0147	8.0070	7.59681989
32	2.0810	1.3483	9.0096	8.0051	7.59682922
64	2.0806	1.3482	9.0057	8.0033	7.59683217
128	2.0804	1.3482	9.0031	8.0019	7.59683295
256	2.0802	1.3481	9.0017	8.0010	7.59683315

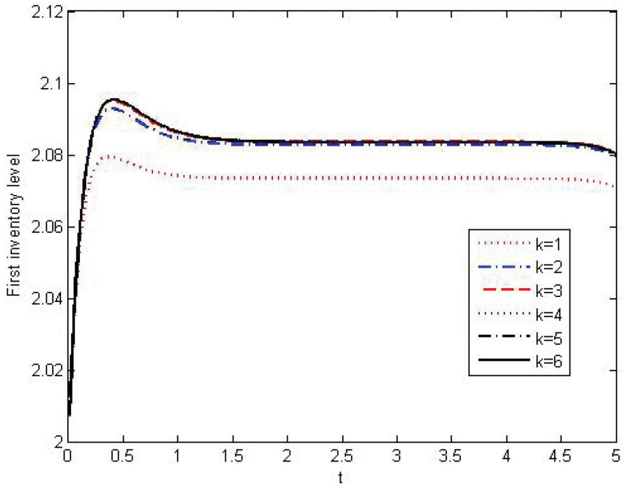

FIGURE 1. First inventory level against time with linear demand rates and $m = 256$ for $k = 1,2,3,4,5,6$ quasilinearization iterations

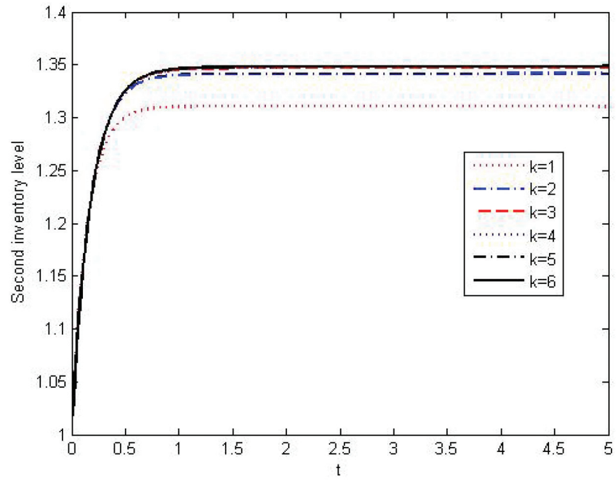

FIGURE 2. Second inventory level against time with linear demand rates and $m = 256$ for $k = 1,2,3,4,5,6$ quasilinearization iterations

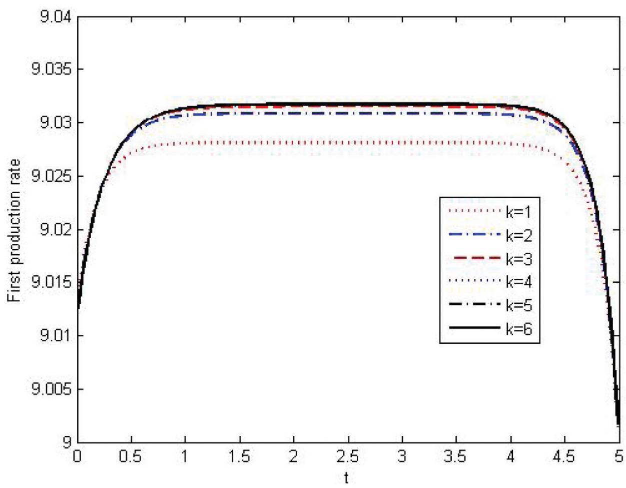

FIGURE 3. First production rate against time with linear demand rates and $m = 256$ for $k = 1,2,3,4,5,6$ quasilinearization iterations

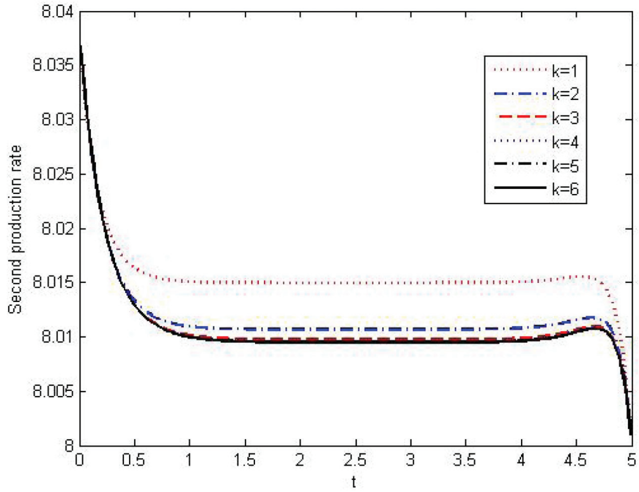

FIGURE 4. Second production rate against time with linear demand rates and $m = 256$ for $k = 1,2,3,4,5,6$ quasilinearization iterations

TABLE 2. Result obtained from El-Gohary and Elsayed (2008) for application with linear demand rates

Demand rates	$y_1^*(t_f)$	$y_2^*(t_f)$	$v_1^*(t_f)$	$v_2^*(t_f)$	J^*
Linear	2.08	1.35	9	8	7.6

indicates that the approximated cost function converges to the true cost function as we increase the resolution of the Haar wavelet. Figures 1 and 2 also suggest that the optimal inventory levels increase over time. Figures 3 and 4 show how production rates were optimized and tended to their goal rates at the end of the planning horizon period.

El-Gohary and Elsayed (2008) reduced the same application problem into a system of differential equations according to the Pontryagin principle and numerically solved this system with the Runge-Kutta method to obtain the values as in Table 2. However, the indirect method used by El-Gohary and Elsayed (2008) has a drawback because the system contains co-state variables, which are not physical entities. Moreover, if the final state is fixed, the indirect method needs to solve a two-point boundary value problem.

Although we have considerd $m = 256$ in our computation, but Table 1 shows that the usage of $m = 32$ is enough to approximate the optimal cost function and trajectory variables to the same accuracy as that obtained in Table 2.

CONCLUSION

In this study, we proposed a new numerical method to solve nonlinear optimal control problems with state and control inequality constraints. Our approach uses the quasilinearization method and the operational matrix of the Haar wavelet to convert the nonlinear optimal control problem into a sequence of linear quadratic programming problems that are constrained and time-varying. The method has been tested on optimal control problems that are constrained and nonlinear in two-dimensional state spaces with two controllers. In particular, we apply it to the two-item inventory model with stock-dependent deterioration rates and linear demand rates. In addition, the results are compared with the existing numerical solutions. Our method is simple and required fewer collocation points to achieve the same accuracy as the existing numerical solution. By increasing the Haar wavelet resolution, we can always improve the accuracy of the states, controls and cost.

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